

Series solution for continuous population models for single and interacting species by the homotopy analysis method

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Abstract

The homotopy analysis method (HAM) is used to find approximate analytical solutions of continuous population models for single and interacting species. The homotopy analysis method contains the auxiliary parameter \hbar , which provides us with a simple way to adjust and control the convergence region of series solution. The solutions are compared with the numerical results obtained using NDSolve, an ordinary differential equation solver found in the Mathematica package and a good agreement is found. Also the solutions are compared with the available analytic results obtained by other methods and more accurate and convergent series solution found. The convergence region is also computed which shows the validity of the HAM solution. This method is reliable and manageable.

Keywords : Homotopy analysis method; Convergence-controller parameter; Nonlinear differential equations; Logistic growth; Predator-prey models.

1 Introduction

The homotopy analysis method (HAM) is an analytical technique for solving nonlinear differential equations. Proposed by Liao in 1992 [1], the technique is superior to the traditional perturbation methods in that it leads to convergent series solutions of strongly nonlinear problems, independent of any small or large physical parameter associated with

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the problem [2]. The HAM provides a more viable alternative to nonperturbation techniques such as the Adomian decomposition method (ADM) [3, 4, 5] and other techniques that cannot guarantee the convergence of the solution series and may be only valid for weakly nonlinear problems [2]. We note here that He's homotopy perturbation method (HPM) [6, 7] is only a special case of the HAM [8]. Indeed Liao [9] makes a compelling case that the Adomian decomposition method, the Lyapunov artificial small parameter method and the δ -expansion method are nothing but special cases of the HAM. In recent years; this method has been successfully employed to solve many in science and engineering such as the viscous flows of non-Newtonian fluids [10, 11, 12, 13, 14, 15, 16], the KdV -type equations [17, 18, 19, 20, 21], Glauert-jet problem [22], Burgers-Huxley equation [23], time-dependent Emden-Fowler type equations [24], differential-difference equation [25], the MHD Falkner-Skan flow [26], multiple solutions of nonlinear problems [27, 28, 29, 30], Schrödinger equations [31], two-point nonlinear boundary value problems [32], a new technique of using homotopy analysis method for solving high-order nonlinear differential equations [33], and more other applications [34, 35, 36, 37, 38, 39, 40].

The main aim of this paper is to present applications of the homotopy analysis method (HAM) to four non-linear biological problems. The first problem is a logistic growth model in a population whereas the second one is a prey-predator model: Lotka-Volterra system. The third problem is a simple 2-species Lotka-Volterra competition model, and the fourth one is a prey-predator model with limit cycle periodic behavior.

2 The homotopy analysis method

To illustrate the basic concept of the HAM, we consider the following general nonlinear differential equation

$$N[u(\tau)] - f(\tau) = 0 \quad (2.1)$$

where N is a nonlinear operator, τ denote independent variable, $f(\tau)$ is a known analytic function, and $u(\tau)$ is an unknown function. By means of generalizing the traditional homotopy method, Liao [1] constructs the so-called zero-order deformation equation.

$$(1 - q)L[\Phi(\tau; q) - u_0(\tau)] = q\hbar[N[\Phi(\tau; q) - f(\tau)]] \quad (2.2)$$

where $q \in [0, 1]$ denote the so-called embedding parameter. $\hbar \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator. The homotopy analysis method is based on a kind of continuous mapping $u(\tau) \rightarrow \Phi(\tau, q)$, $\Phi(\tau, q)$ is an unknown function and $u_0(\tau)$ is an initial guess of $u(\tau)$. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, Eq. (2.2) becomes

$$\Phi(\tau; 0) = u_0(\tau), \Phi(\tau; 1) = u(\tau) \quad (2.3)$$

respectively. Thus as q increases from 0 to 1, the solution $\Phi(\tau, q)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. In topology, this kind of variation is called deformation. Eq. (2.2) construct the homotopy $\Phi(\tau, q)$, and Eq. (2.2) is called zero-order deformation equation. Having the freedom to choose the auxiliary parameter \hbar , the initial approximation $u_0(\tau)$, and the auxiliary linear operator L , we can assume that all of them are properly chosen so that the solution $\Phi(\tau, q)$ of the zero-order deformation Eq. (2.2) exists for $0 \leq q \leq 1$. Expanding $\Phi(\tau, q)$ in Taylor series with respect to q , one has

$$\Phi(\tau, q) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)q^m, \quad (2.4)$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \Phi(x, t; q)}{\partial q^m} \Big|_{q=0} \quad (2.5)$$

Assume that the auxiliary parameter \hbar , the initial approximation $u_0(\tau)$ and the auxiliary linear operator L are so properly chosen that the series Eq. (2.4) converges at $q = 1$ and

$$\Phi(\tau, 1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \quad (2.6)$$

which must be one of solutions of the original nonlinear equation, as proved by Liao [1]. As $\hbar = -1$, Eq. (2.2) becomes

$$(1 - q)L[\Phi(\tau; q) - u_0(\tau)] + q[N[\Phi(\tau; q) - f(\tau)]] = 0 \quad (2.7)$$

which is used mostly in the homotopy-perturbation method [7]. According to the definition Eq. (2.5), the governing equation and corresponding initial condition of $u_m(x, t)$ can be deduced from the zero-order deformation Eq. (2.2). Define the vector

$$\vec{u}_n(x, t) = u_0(x, t), u_1(x, t), u_2(x, t), \dots, u_n(x, t).$$

Differentiating Eq. (2.2) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called m th-order deformation equation:

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar R(\vec{u}_{m-1}), \quad (2.8)$$

where

$$R(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} [N[\Phi(\tau; q) - r(\tau)]]}{\partial q^{m-1}} \Big|_{q=0}, \quad (2.9)$$

and

$$\begin{cases} \chi_m = 0, & m \leq 1, \\ \chi_m = 1, & \text{otherwise} \end{cases} \quad (2.10)$$

It should be emphasized that $u_m(\tau)$ for $m \geq 1$ is governed by the linear Eq. (2.8) with linear boundary conditions that come from the original problem, which can be solved by the symbolic computation software such as Mathematica, Maple and Matlab.

3 Case studies

3.1 Case study 1

We first consider the logistic growth in a population as a single species model to be governed by [41]

$$\frac{dN}{dt} = rN(1 - N/K), \quad (3.11)$$

where r and K are positive constants. Here $N = N(t)$ represents the population of the species at time t , and $r(1 - N/K)$ and is the per capita growth rate, and K is the carrying

capacity of the environment. Let $u(\tau) = N(t)/K$, $\tau = rt$.
Then Eq. (3.11) becomes

$$\frac{dN}{d\tau} = u(1 - u). \quad (3.12)$$

If $N(0) = N_0$, then $u(0) = N_0/K$. Therefore, the analytical solution of Eq. (3.12) is

$$u(\tau) = \frac{1}{1 + (K/N_0 - 1)e^{-t}} \quad (3.13)$$

For HAM solution we choose the linear operator

$$L[\Phi(\tau; q)] = \frac{\partial \Phi(\tau; q)}{\partial \tau}, \quad (3.14)$$

with the property $L[c_1] = 0$, where c_1 is a constant. Using initial approximation

$$u_0(\tau) = N_0/K, \quad (3.15)$$

We define a nonlinear operator as

$$N[\Phi(\tau; q)] = \frac{\partial \Phi(\tau; q)}{\partial \tau} - \Phi(\tau; q) + (\Phi(\tau; q))^2,$$

where

$$R_m(\vec{u}_{m-1}(\tau)) = \frac{\partial u_{m-1}(\tau)}{\partial \tau} - u_{m-1}(\tau) + \sum_{i=0}^{m-1} u_i(\tau)u_{m-1-i}(\tau) \quad (3.16)$$

Now the solution of the m th- order deformation Eq. (2.8) for $m \geq 1$ becomes

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \hbar \int R(\vec{u}_{m-1}(\tau))d\tau + c_1.$$

where the constant of integration c_1 is determined by the initial conditions

$$u_m(0) = 0, \quad (3.17)$$

and determined by Eq. (2.10). We now obtain by HAM

$$u_1(\tau) = \hbar \left(-\frac{N_0}{K} + \frac{N_0^2}{K^2} \right) \tau,$$

$$u_2(\tau) = \left(-\frac{\hbar N_0}{K} - \frac{\hbar^2 (K - N_0) N_0}{K^2} + \frac{\hbar N_0^2}{K^2} \right) \tau + \frac{\hbar^2 (K - 2N_0)(K - N_0) N_0}{2K^3} \tau^2,$$

$u_m(\tau)$ ($m = 3, 4, \dots$) can be calculated similarly. Then the series solution expression by HAM can be written in the form

$$U_M(\tau, \hbar) = \sum_{i=0}^M u_i(\tau, \hbar), \quad (3.18)$$

Recall that

$$u(\tau) = \lim_{M \rightarrow \infty} U_M, \quad (3.19)$$

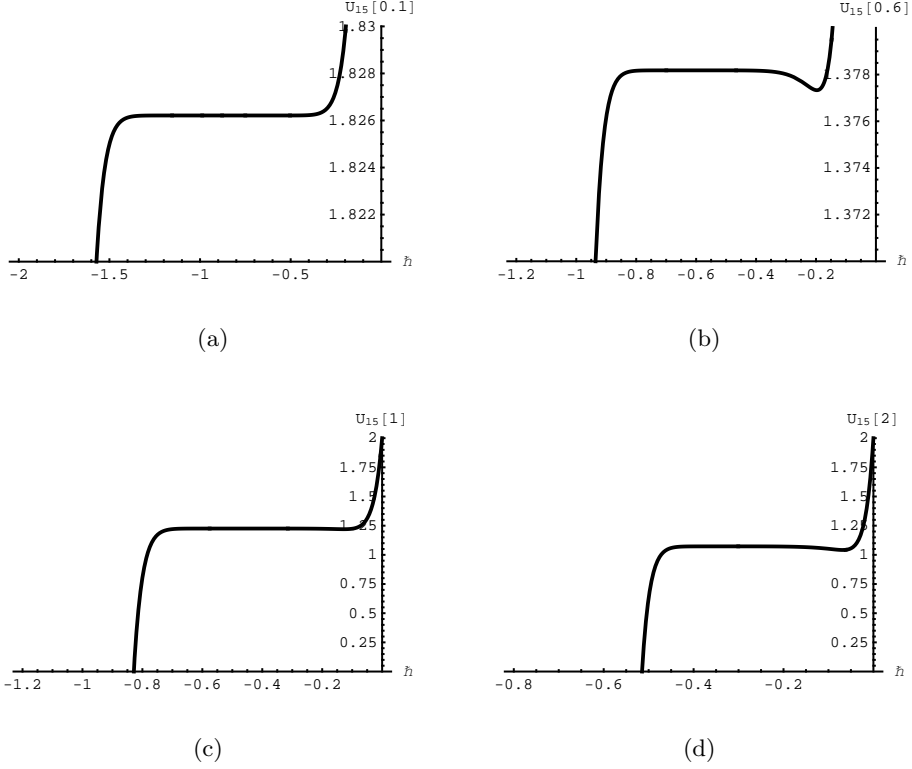


Figure 1: The \hbar curves of (a) $U_{15}(0.1)$, (b) $U_{15}(0.6)$, (c) $U_{15}(1)$, and (d) $U_{15}(2)$ by HAM for logistic growth Eq. (3.12)

For numerical comparison we take $N_0 = 2$ and $K = 1$ [42] therefore Eq. (3.13) becomes

$$u(\tau) = \frac{2}{2 - \exp -\tau}, u(0) = 2. \quad (3.20)$$

Figs 1a, 1b, 1c and 1d show the \hbar -curves obtained from the $U_{15}(\tau)$ HAM approximation solutions of Eq. (3.12) at different values of τ . These figures show the interval of \hbar which the value of $U_{15}(\tau)$ is constant at certain τ , we choose the horizontal line parallel to x -axis as valid \hbar region which provides us with a simple way to adjust and control the convergence region of series solution, from this figures $\hbar = -1$ is valid only for small τ ($0 < \tau < 0.375$), but $-0.44 \leq \hbar \leq -0.15$ are valid for larger ($0 < \tau < 2$). To demonstrate the efficiency of HAM for solving logistic growth model Eq. (3.12) we plot many figures.

Figs 2 and 3 show a very good approximation to the analytical solution of logistic growth model in the time interval ($0 < \tau < 2$) by using only 6 terms of the series given by Eq. (3.18) which indicates that the speed of convergence of HAM at $\hbar = -0.38$ is very fast. In addition, a better approximation to the exact solution can be achieved by adding new terms to this series as shown in figs 2 and 3. Figs 4 and 5 show that $\hbar = -1$ give good approximation for only ($\tau < 0.375$) as the same result obtained by HPM [42],[43] and ADM [44] using 9 terms of the series. As mention before HPM and ADM are special cases of HAM ($\hbar = -1$). The comparison between $U_8(\tau)$ of HAM at $\hbar = -0.38$ and HPM (HAM $\hbar = -1$) shows the importance of \hbar .

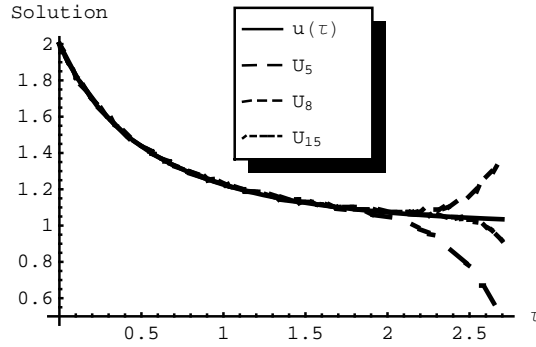


Figure 2: Comparison between $U_5(\tau)$, $U_8(\tau)$ and $U_{15}(\tau)$ of HAM and $u(\tau)$ for logistic growth model Eq. (3.12) with $\hbar = -0.38$.

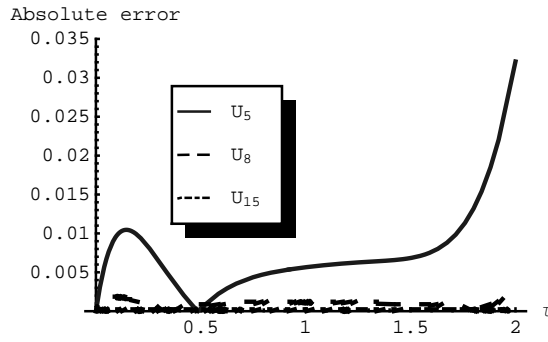


Figure 3: The absolute errors of $U_5(\tau)$, $U_8(\tau)$ and $U_{15}(\tau)$ of HAM for logistic growth model Eq. (3.12) with $\hbar = -0.38$.

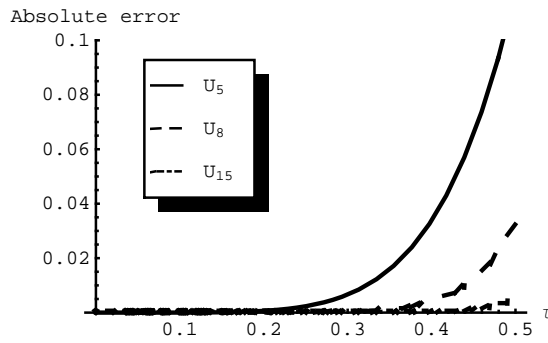


Figure 4: The absolute errors of $U_5(\tau)$, $U_8(\tau)$ and $U_{15}(\tau)$ of HAM and $u(\tau)$ for logistic growth model Eq. (3.12) at $0 < \tau < 0.5$ with $\hbar = -1$.

3.2 Case study 2

We consider the Predator-Prey Models: Lotka-Volterra systems as an interacting species model to be governed by [41]

$$\frac{dN}{dt} = N(a - bP), \quad \frac{dP}{dt} = P(cN - d). \quad (3.21)$$

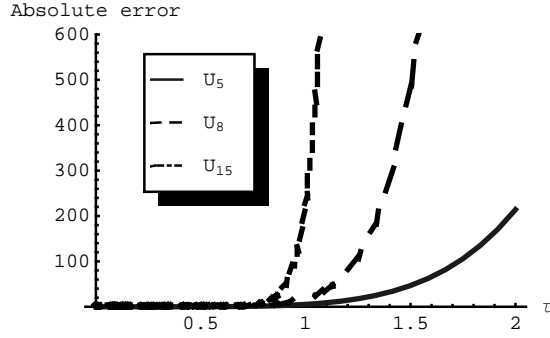


Figure 5: The absolute errors of $U_5(\tau)$, $U_8(\tau)$ and $U_{15}(\tau)$ of HAM and $u(\tau)$ for logistic growth model Eq. (3.12) at $0 < \tau < 2$ with $\hbar = -1$.

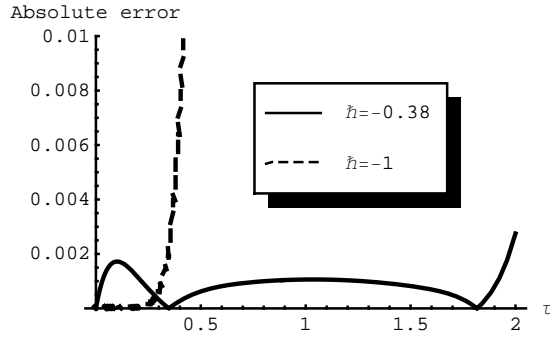


Figure 6: Comparison between the errors of $U_8(\tau)$ at $\hbar = -1$ and $U_8(\tau)$ at $\hbar = -0.38$ of HAM for logistic growth model Eq. (3.12) at $0 < \tau < 2$.

where a , b , c and d are constants. Here $N = N(t)$ is the prey population and $P = P(t)$ that of the predator at time t . Let $u(\tau) = \frac{cN(t)}{d}$, $v(\tau) = \frac{bP(t)}{a}$, $\tau = at$, $\alpha = d/a$, then Eq. (3.21) becomes

$$\frac{du}{d\tau} = u(1-v), \quad \frac{dv}{d\tau} = \alpha v(u-1). \quad (3.22)$$

According to HAM, we choose the auxiliary linear operator

$$L[\Phi_1(\tau; q)] = \frac{\partial \Phi_1(\tau; q)}{\partial \tau}, \quad L[\Phi_2(\tau; q)] = \frac{\partial \Phi_2(\tau; q)}{\partial \tau}, \quad (3.23)$$

with the property $L[c_i] = 0$, where c_i ($i = 1, 2$) are integral constants. Furthermore, Eq. (3.22) suggest that a system of nonlinear operators as

$$N_1[\Phi_1(\tau; q)] = \frac{\partial \Phi_1}{\partial \tau} - \Phi_1 + \Phi_1 \Phi_2,$$

$$N_2[\Phi_2(\tau; q)] = \frac{\partial \Phi_2}{\partial \tau} - \alpha \Phi_2 \Phi_1 + \alpha \Phi_2,$$

with

$$R_{1,m}(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial \tau} - u_{m-1} + \sum_{i=0}^{m-1} u_i u_{m-1-i},$$

$$R_{2,m}(\vec{v}_{m-1}) = \frac{\partial v_{m-1}}{\partial \tau} - \alpha \sum_{i=0}^{m-1} u_i v_{m-1-i} + \alpha v_{m-1}$$

Now the solution of the m th-order deformation Eq. (2.8) for $m \geq 1$ become

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \hbar L^{-1}[R_{1,m}(\vec{u}_{m-1})] = \chi_m u_{m-1}(\tau) + \hbar \int R_{1,m}(\vec{u}_{m-1}) d\tau + c_1, \quad (3.24)$$

and

$$v_m(\tau) = \chi_m v_{m-1}(\tau) + \hbar L^{-1}[R_{2,m}(\vec{v}_{m-1})] = \chi_m v_{m-1}(\tau) + \hbar \int R_{2,m}(\vec{v}_{m-1}) d\tau + c_2, \quad (3.25)$$

where the coefficients c_1 and c_2 are determined respectively by the initial conditions for $m \geq 1$

$$u_m(0) = 0, \quad v_m(0) = 0. \quad (3.26)$$

We choose the initial approximation [42]

$$\begin{aligned} u_0(\tau) &= u(0) = 1.3, \\ v_0(\tau) &= v(0) = 0.6. \end{aligned}$$

We now successively obtain

$$\begin{aligned} u_1(\tau) &= -0.52\hbar\tau, \\ v_1(\tau) &= -0.18\alpha\hbar\tau \end{aligned}$$

$u_m(\tau)$ and $v_m(\tau)$ ($m = 2, 3, \dots$) can be calculated similarly. Then the series solution expression by HAM can be written in the form

$$U_M(\tau, \hbar) = \sum_{m=0}^M u_m(\tau, \hbar), \quad (3.27)$$

$$V_M(\tau, \hbar) = \sum_{m=0}^M v_m(\tau, \hbar). \quad (3.28)$$

For numerical comparison, we take $\alpha = \frac{m^2-1}{0}$ [42]. Figs 7a and 7b show the \hbar -curves obtained from the $U_{15}(\tau)$ and $V_{15}(\tau)$ HAM approximation solutions of Eq. (3.22) at different values of τ . These figures show that $\hbar = -1$ is the proper value for convergent the series Eq. (3.27) and Eq. (3.28). Figs 8 and 9 show the comparison between the $U_4(\tau)$, $U_{10}(\tau)$ and $U_{15}(\tau)$ HAM solutions of the system Eq. (3.22) with the numerical solutions and the comparison between , and HAM solutions of the system Eq. (3.28) with the numerical solutions respectively. We obtain these numerical solutions using NDSolve, an ordinary differential equation solver found in the Mathematica package. It is clear from both of the figures that there is a very close agreement between the solutions for u (prey population), v (predator population) and numerical solution. A better approximation to the exact solution can be achieved by adding new terms to this series as shown in figs 10 and 11. The obtained results in this example are the same results of HPM [42], [43] and ADM [44].

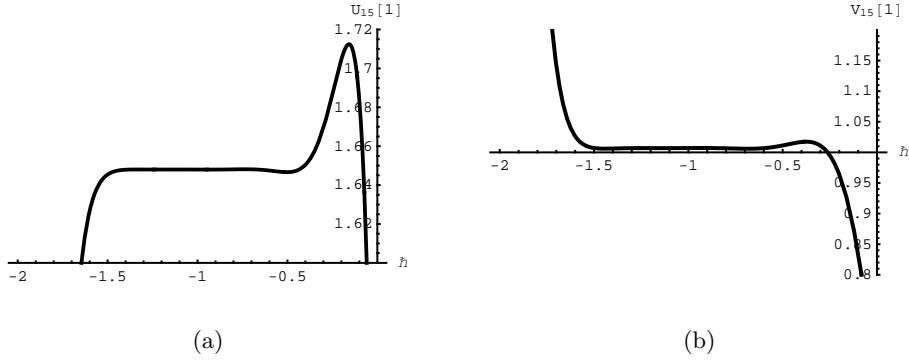


Figure 7: The \hbar curves of (a) $U_{15}(1)$, (b) $V_{15}(1)$ by HAM for Predator-Prey Models at $\alpha = 1$ Eq.(3.22)

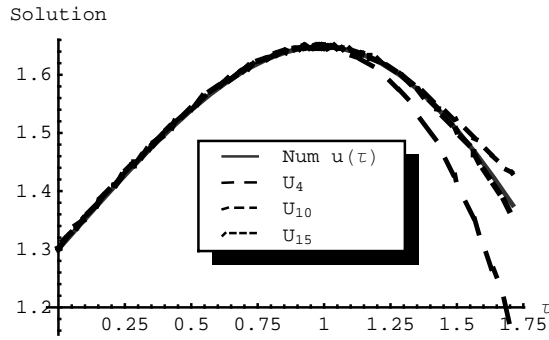


Figure 8: Comparison between $U_4(\tau)$, $U_{10}(\tau)$ and $U_{15}(\tau)$ of HAM and numerical solution $u(\tau)$ for Predator-Prey Models Eq. (3.22) with $\alpha = 1$ and $\hbar = -1$.

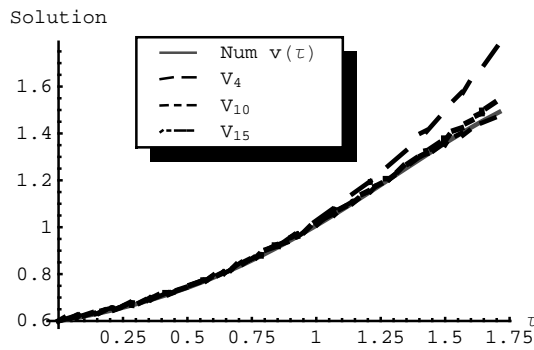


Figure 9: Comparison between $V_4(\tau)$, $V_{10}(\tau)$ and $V_{15}(\tau)$ of HAM and numerical solution $v(\tau)$ for Predator-Prey Models Eq. (3.22) with $\alpha = 1$ and $\hbar = -1$.

3.3 Case study 3

We consider the simple 2-species Lotka-Volterra competition model with each species N_1 and N_2 having logistic growth in the absence of the other. Inclusion of logistic growth in the Lotka-Volterra systems makes them much more realistic but to highlight the principle

we consider the simpler model which nevertheless reflects many of the properties of more complicated models, particularly as regards stability. We thus consider the system [41]

$$\frac{dN_1}{dt} = r_1 N_1 \left[1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1} \right], \quad \frac{dN_2}{dt} = r_2 N_2 \left[1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_2} \right]. \quad (3.29)$$

where $r_1, K_1, r_2, K_2, b_{12}$ and b_{21} are all positive constants and the r 's are the linear birth rates and the K 's are the carrying capacities. The b_{12} and b_{21} measure the competitive effect of N_2 on N_1 and N_1 on N_2 respectively, they are generally not equal. Let

$$u = \frac{N_1}{K_1}, \quad v = \frac{N_2}{K_2}, \quad \tau = r_1 t, \quad \rho = \frac{r_2}{r_1}, \quad a = b_{12} \frac{K_2}{K_1}, \quad b = b_{21} \frac{K_1}{K_2},$$

The system given by Eq. (3.29) becomes

$$\frac{du}{d\tau} = u(1 - u - \alpha v), \quad \frac{dv}{d\tau} = \rho v(1 - v - bu). \quad (3.30)$$

According to HAM, we choose the auxiliary linear operator Eq. (3.23). Furthermore, Eq. (3.30) suggest that we define a system of nonlinear operators as

$$N_1[\Phi_1(\tau; q)] = \frac{\partial \Phi_1}{\partial \tau} - \Phi_1 + \Phi_1^2 + a\Phi_1\Phi_2,$$

$$N_2[\Phi_2(\tau; q)] = \frac{\partial \Phi_2}{\partial \tau} - \rho\Phi_2 + \rho\Phi_2^2 + \rho b\Phi_1\Phi_2,$$

with

$$R_{1,m}(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial \tau} - u_{m-1} + \sum_{i=0}^{m-1} u_i u_{m-1-i} + a \sum_{i=0}^{m-1} u_i v_{m-1-i},$$

$$R_{2,m}(\vec{v}_{m-1}) = \frac{\partial v_{m-1}}{\partial \tau} - \rho v_{m-1} + \rho \sum_{i=0}^{m-1} v_i v_{m-1-i} + \rho b \sum_{i=0}^{m-1} u_i v_{m-1-i}$$

Now the solution of the m th-order deformation Eq. (2.8) for $m \geq 1$ become

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \hbar L^{-1}[R_{1,m}(\vec{u}_{m-1})] = \chi_m u_{m-1}(\tau) + \hbar \int R_{1,m}(\vec{u}_{m-1}) d\tau + c_1, \quad (3.31)$$

and

$$v_m(\tau) = \chi_m v_{m-1}(\tau) + \hbar L^{-1}[R_{2,m}(\vec{v}_{m-1})] = \chi_m v_{m-1}(\tau) + \hbar \int R_{2,m}(\vec{v}_{m-1}) d\tau + c_2, \quad (3.32)$$

where the coefficients c_1 and c_2 are determined respectively by the initial conditions Eq. (2.8) for $m \geq 1$. We choose the initial approximation [42]

$$u_0(\tau) = u(0) = 1, \\ v_0(\tau) = v(0) = 1$$

We now successively obtain

$$u_1(\tau) = a\hbar\tau, \\ v_1(\tau) = \hbar\tau(1 + b - \rho)$$

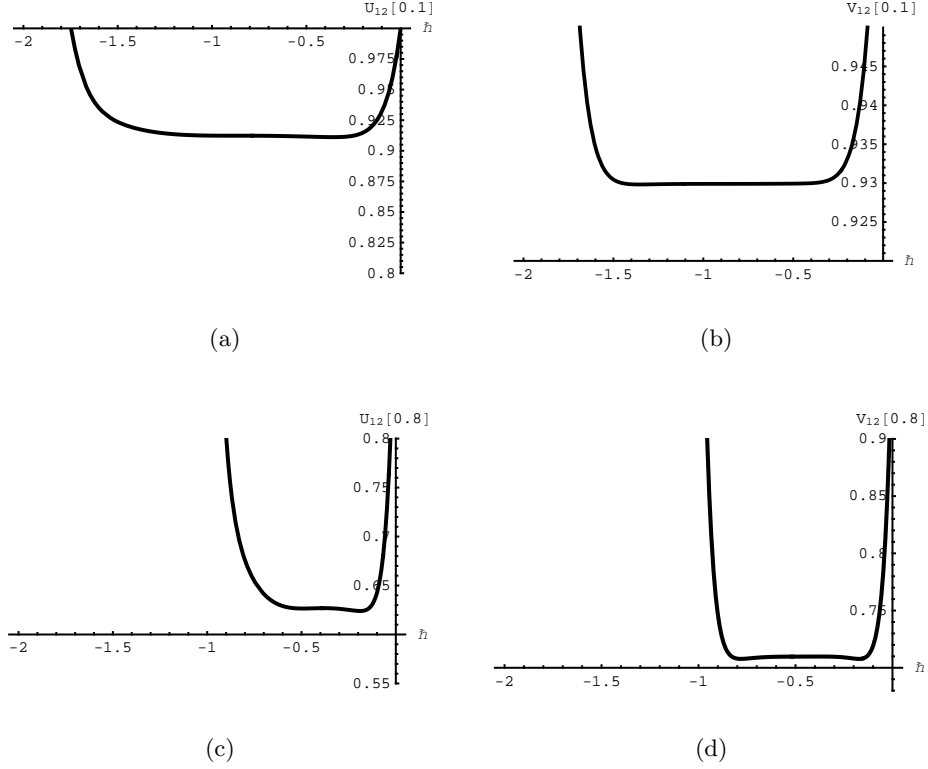


Figure 10: The \hbar curves of (a) $U_{12}(0.1)$, (b) $V_{12}(0.1)$, (c) $U_{12}(0.8)$, and (d) $V_{12}(0.8)$ by HAM for Eq. (3.30)

$u_m(\tau)$ and $v_m(\tau)$ ($m = 2, 3, \dots$) can be calculated similarly. Then the series solution expression by HAM can be written in the form Eq. (3.27) and Eq. (3.28)

For numerical comparison, we take $a = 1$, $\rho = 1$ and $b = 0.8$ [42]. Figs 10a and 10b show the \hbar -curves obtained from the $U_{12}(\tau)$ HAM approximation solutions of Eq. (3.30) at different values of τ . These figures show that $\hbar = -1$ is valid only for small τ ($0 < \tau < 0.25$), but around $\hbar = -0.5$ are valid for larger τ ($0 < \tau < 1.25$). Figs 11 and 12 show the absolute errors of $U_4(\tau)$ and $U_{12}(\tau)$ of HAM and the absolute errors of $V_4(\tau)$ and $V_{12}(\tau)$ of HAM of the system Eq. (3.30) using $\hbar = -0.5$ with respect to the numerical solution respectively. We obtain these numerical solutions using NDSolve. It is clear from both of the figures that there is a very close agreement between the solutions for u , v and numerical solutions. Also a better approximation to the exact solution can be achieved by adding new terms to this series using $\hbar = -0.5$, which indicates that the speed of convergence of HAM at $\hbar = -0.5$ is very fast. Figs 13 and 14 show that $\hbar = -1$ gives good approximation for only ($0 < \tau < 0.25$) as the same result obtained by HPM [42] using 5 terms of the series. The comparison between $U_4(\tau)$ of HAM at $\hbar = -0.5$ and HPM ($\hbar = -1$ HAM) shows the importance of \hbar .

3.4 Case study 4

We consider a prey-predator model with limit cycle periodic behavior [41]:

$$\frac{dN}{dt} = N[r(1 - \frac{N}{K}) - \frac{kP}{N+D}], \quad \frac{dP}{dt} = P[r(s(1 - \frac{hP}{N})]. \quad (3.33)$$

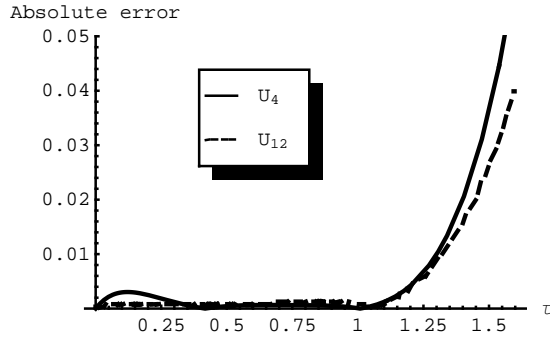


Figure 11: The absolute errors of $U_4(\tau)$ and $U_{12}(\tau)$ of HAM with $\hbar = -0.5$ for Eq. (3.30).

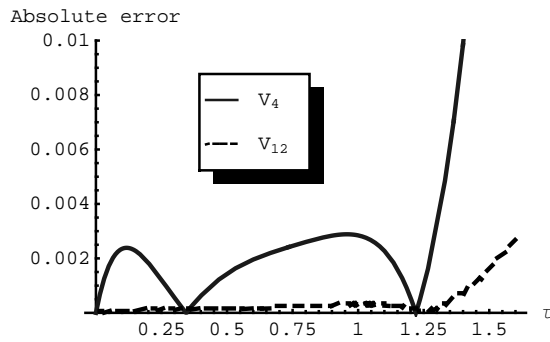


Figure 12: The absolute errors of $V_4(\tau)$ and $V_{12}(\tau)$ of HAM with $\hbar = -0.5$ for Eq. (3.30).

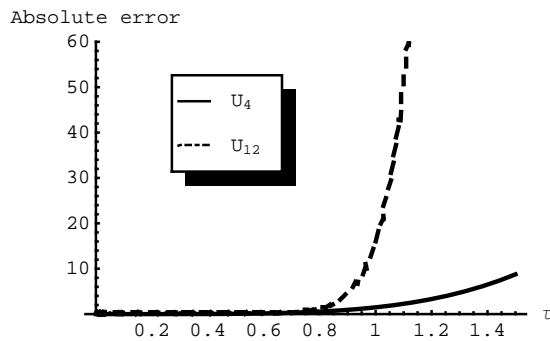


Figure 13: The absolute errors of $U_4(\tau)$ and $U_{12}(\tau)$ of HAM at $\hbar = -1$ for Eq. (3.30).

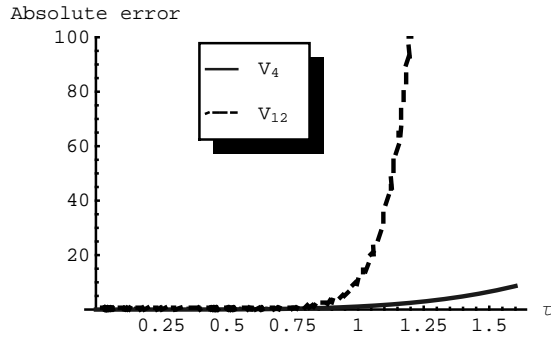


Figure 14: The absolute errors of $V_4(\tau)$ and $V_{12}(\tau)$ of HAM at $\hbar = -1$ for Eq. (3.30).

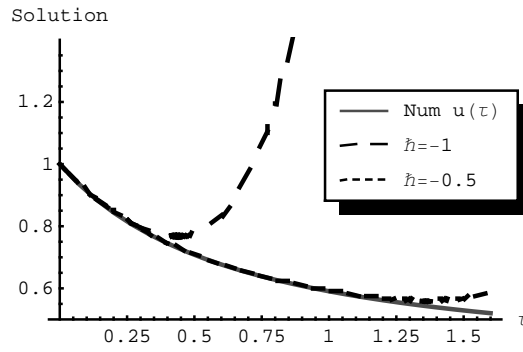


Figure 15: Comparison between $U_4(\tau)$ at $\hbar = -1$ and $U_4(\tau)$ at $\hbar = -0.5$ of HAM for Eq. (3.30).

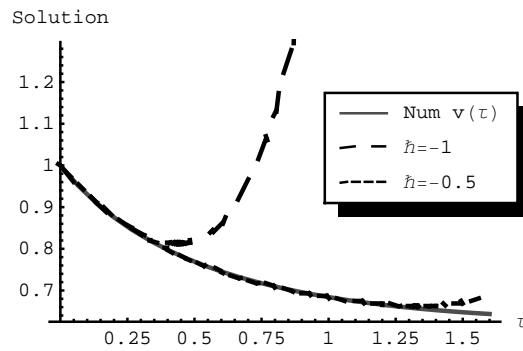


Figure 16: Comparison between $V_4(\tau)$ at $\hbar = -1$ and $V_4(\tau)$ at $\hbar = -0.5$ of HAM for Eq. (3.30).

where r, K, r, D, s and h are all positive constants. Let

$$u = \frac{N}{K}, \quad v = \frac{hP}{K}, \quad \tau = rt, \quad a = \frac{k}{hr}, \quad b = \frac{s}{r}, \quad d = \frac{D}{K},$$

The system given by Eq. (3.33) becomes

$$\frac{du}{d\tau} = u(1-u) - \frac{auv}{u+d}, \quad \frac{dv}{d\tau} = bv(1 - \frac{v}{u}). \quad (3.34)$$

According to HAM, we choose the auxiliary linear operator Eq. (3.23). Furthermore, Eq. (3.34) suggest that we define a system of nonlinear operators as

$$N_1[\Phi_1(\tau; q)] = \frac{\partial \Phi_1}{\partial \tau} - \Phi_1 + \Phi_1^2 + \frac{a\Phi_1\Phi_1}{\Phi_1+d},$$

$$N_2[\Phi_2(\tau; q)] = \frac{\partial \Phi_2}{\partial \tau} - b\Phi_2 + \frac{b\Phi_2^2}{\Phi_1},$$

and from Eq. (2.9), we get

$$R_{1,1}(\vec{u}_0) = \frac{\partial u_0}{\partial \tau} - u_0 + u_0^2 + \frac{au_0v_0}{d+u_0},$$

$$R_{1,1}(\vec{v}_0) = \frac{\partial v_0}{\partial \tau} - bv_0(1 - \frac{v_0}{u_0}),$$

$$R_{1,2}(\vec{u}_1) = \frac{\partial u_1}{\partial \tau} - u_1 + 2u_0v_0 + a\frac{u_1v_0+u_0v_1}{d+u_0} - a\frac{u_1v_0u_0^2}{(d+u_0)^2},$$

$$R_{1,2}(\vec{v}_1) = \frac{\partial v_1}{\partial \tau} - bv_1 + \frac{2bv_0v_1}{u_0} - \frac{bv_0^2u_1}{u_0^2},$$

⋮

Now the solution of the m th-order deformation Eq. (2.8) for $m \geq 1$ become

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \hbar L^{-1}[R_{1,m}(\vec{u}_{m-1})] = \chi_m u_{m-1}(\tau) + \hbar \int R_{1,m}(\vec{u}_{m-1})d\tau + c_1, \quad (3.35)$$

and

$$v_m(\tau) = \chi_m v_{m-1}(\tau) + \hbar L^{-1}[R_{2,m}(\vec{v}_{m-1})] = \chi_m v_{m-1}(\tau) + \hbar \int R_{2,m}(\vec{v}_{m-1})d\tau + c_2, \quad (3.36)$$

where the coefficients c_1 and c_2 are determined respectively by the initial conditions Eq. (3.26) for $m \geq 1$. We choose the initial approximation [42]

$$u_0(\tau) = u(0) = 1.3,$$

$$v_0(\tau) = v(0) = 1.2$$

We now successively obtain

$$u_1(\tau) = (0.39 + \frac{1.56a}{1.3+d})\hbar\tau,$$

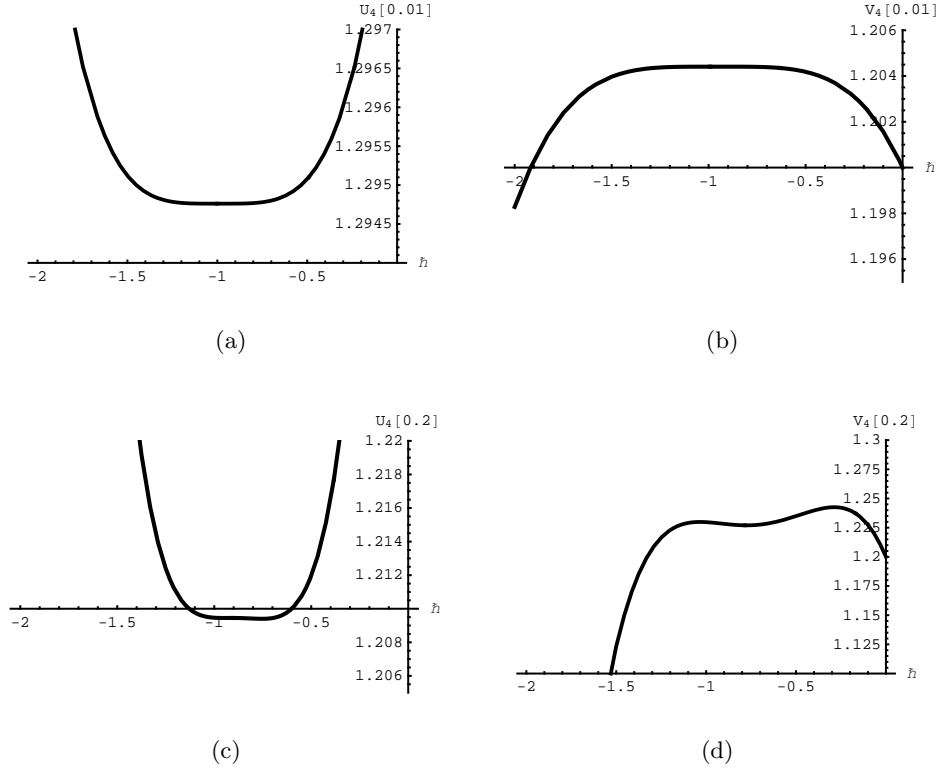


Figure 17: The \hbar curves of (a) $U_4(0.01)$, (b) $V_4(0.01)$, (c) $U_4(0.2)$, and (d) $V_4(0.2)$ by HAM for Eq. (3.34)

$$v_1(\tau) = -0.0923077b\hbar\tau$$

$u_m(\tau)$ and $v_m(\tau)$ ($m = 2, 3, \dots$) can be calculated similarly. Then the series solution expression by HAM can be written in the form Eq. (3.27) and Eq. (3.28).

For numerical comparison we take $a = 1$, $d = 10$ and $b = 5$ [42].

Figures 17a and 17b show the \hbar -curves obtained from the $U_4(\tau)$ and $V_4(\tau)$ HAM approximation solutions of Eq. (3.34) at different values of τ . These figures show that $\hbar = -1$ is valid only for small τ ($0 < \tau < 0.6$) for $U_4(\tau)$ and ($0 < \tau < 0.12$) for $V_4(\tau)$, but $\hbar = -0.85$ is valid for larger τ ($0 < \tau < 0.9$) for $U_4(\tau)$ and ($0 < \tau < 0.23$) for $V_4(\tau)$. Figs 18 and 19 show a very good approximation to the numerical solution of Eq. (3.34) using only 5 terms of the series $U_4(\tau)$ and $V_4(\tau)$ given by Eq. (3.27) and Eq. (3.28) respectively with $\hbar = -0.85$ and $\hbar = -1$ which indicates that the series solution of HAM at $\hbar = -0.85$ is more accurate than series solution of HAM at $\hbar = -1$. We obtain these numerical solutions using NDSolve. A better approximation for larger time interval can be achieved by adding new terms to the series using $\hbar = -0.85$ as shown in figs 20 and 21.

4 Conclusions

In this work, the homotopy analysis method (HAM) was used for finding the analytical solutions of continuous population models for single and interacting species. The validity of our solutions is verified by the numerical and other analytic results. We analyzed the convergence of the obtained series solutions, carefully. The convergence analysis elucidates

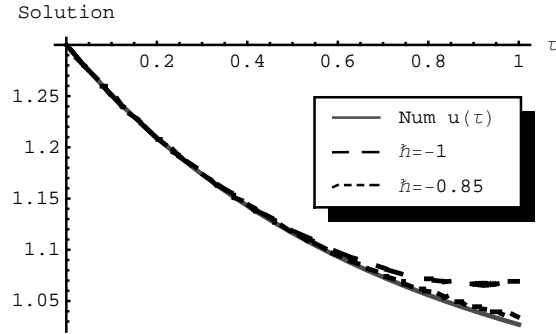


Figure 18: Comparison between $U_4(\tau)$ with $\hbar = -0.85$, $U_4(\tau)$ with $\hbar = -1$ of HAM and numerical solution for Eq. (3.34).

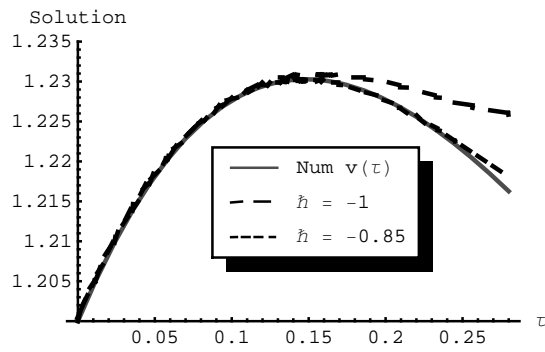


Figure 19: Comparison between $V_4(\tau)$ with $\hbar = -0.85$, $V_4(\tau)$ with $\hbar = -1$ of HAM and numerical solution for Eq. (3.34).

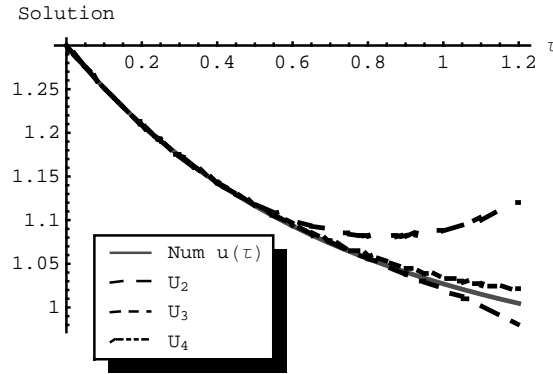


Figure 20: Comparison between $U_2(\tau), U_3(\tau)$ and $U_4(\tau)$ with $\hbar = -0.85$ of HAM and numerical solution for Eq. (3.34).

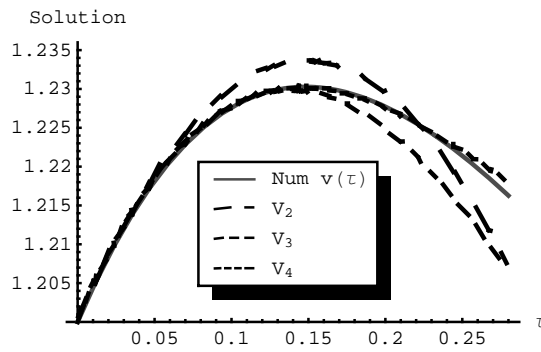


Figure 21: Comparison between $V_2(\tau), V_3(\tau)$ and $V_4(\tau)$ with $\hbar = -0.85$ of HAM and numerical solution for Eq. (3.34).

that the homotopy perturbation method (HPM) and Adomian decomposition method does not give accurate results for our cases study 1, 3 and 4. Unlike perturbation methods, the HAM does not depend on any small physical parameters. Thus, it is valid for both weak and strong nonlinear problems. Besides, different from all other analytic methods, the HAM provides us with a simple way to adjust and control the convergence region of the series solution by means of the auxiliary parameter \hbar . Thus the auxiliary parameter \hbar plays an important role within the frame of the HAM which can be determined by the so called \hbar -curves. The solution obtained by means of the HAM is an infinite power series for appropriate initial approximation, which can be, in turn, expressed in a closed form. Consequently, the present success of the homotopy analysis method for the highly nonlinear problem of continuous population models for single and interacting species verifies that the method is a useful tool for nonlinear problems in science and engineering.

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